AC 2011-2308: PUTTING MATHEMATICS IN CONTEXT: AN INTEGRATIVE APPROACH USING MATLAB

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Putting Mathematics in Context: An Integrative Approach Using Matlab

Introduction

The teaching of foundational engineering mathematics courses can be quite challenging due the abstract nature of the subject matter for freshman and sophomore students. Often it is difficult for the students to see how the material is relevant to their future engineering work, and as a result many students do not become as fully engaged with the course as the instructor may hope for. For some time now, this has been a serious concern for many engineering educators since it is often the case that a large percentage of students do not successfully pass through these foundational courses on their first attempt. This is one of the contributing factors to the large attrition rate which many engineering schools face[1]. One method which has been proven to be effective for increasing student engagement is to place these abstract concepts in a practical context[2,3,4,5]. This way the students can experience the usefulness of these concepts and can relate it to the other subjects which they are learning at the same time.

With the ultimate goal of having students learn the fundamental mathematical concepts presented in the initial years of the program, it is helpful to enable the student see and discover how the material is relevant to the rest of their engineering curriculum. This can provide an extra incentive for the student to engage with the material at hand. Since the vast majority of student learning happens outside of the lecture hall, it is important to create such experiences so the student is motivated to pursue their own studies and efforts in learning. By placing some of the appropriate foundational mathematical concepts in an engineering context and integrating these concepts with their other courses it is possible for such motivation to take place[6,7].

In the fall of 2010, a new component was added to a second-year Advanced Engineering Mathematics (AEM) courses which is a required course in the electrical and computer engineering (ECE) program. This component was based around the use of Matlab, and was introduced for two main reasons. First, it was hoped that by allowing the students to engage with the mathematical concepts in the course through numerical computation their intrinsic interest in the material would increase. Indeed, Matlab enables students to solve more "realistic" engineering problems which are difficult or impossible to solve analytically. Second, the component was designed to be an introduction to Matlab, such that students would have the requisite skills to use Matlab in future courses or to solve their own problems. Prior to this AEM course the students in the ECE program have little or no exposure to Matlab or any other scientific or engineering computational software. This component thus allowed the students to learn a new "tool" for problem solving through the power of numeric analysis.

In the literature, there is a vast amount of work which has described the use of Matlab in the undergraduate engineering curriculum. A large group of this work involves the use of Matlab in freshman level courses which are designed to introduce students to numeric computation and computer programming, for example[8,9]. As well, many educators have published papers describing the use of Matlab to aid in the teaching of specific engineering concepts which often are seen in junior or senior level courses[10,11,12,13]. The current work is unique in that it describes the inclusion of a component on Matlab within a standard AEM course with the intent of

improving the student's interest in the core concepts of the course as well as enabling the students to learn Matlab through the integration of this component with their other ECE courses.

Matlab Component Description

The advanced engineering mathematics course in which this Matlab component was included covers the standard topics of ordinary differential equations (ODEs), Laplace transforms, and complex analysis (including complex numbers, functions of a complex variable, and integration in the complex plane). The component consisted of three self-study modules, two online quizzes, five sets of teaching assistant (TA) office hours (each 2 hours long), and a set of video tutorials.

Matlab Self-Study Modules

The three self-study modules are described in greater detail in Table 1 below, and as an example the third module is presented in Appendix A. These modules were designed to allow the student to work through them on their own time in the engineering faculty's computer labs (which had Matlab R2010a installed). These could also be completed from home through remote access to the faculty's computer network. There were a number of online references provided for the students, most of which are accessible through the MathWorks website[14], as well as the classic Matlab primer[15]. No textbooks were required as it was hoped that the students would learn how to make use of the online help functionality of Matlab as well as the many other online resources.

Table 1. Description of the Three Self-Study Modules

Module Number	Matlab Concepts	Engineering Applications	Relevant ECE Courses
Module #1 Posted in the 1st week of the 13 week semester	• Introduction to the interface and basic syntax • Computations using arrays and matrices	• Solving sets of simultaneous equations • Roots of polynomials	• *Circuit Analysis* (concurrent) • *Advanced Engineering Mathematics*
Module #2 Posted in the 3rd week of the 13 week semester	• Two-dimensional and three-dimensional plots • Writing scripts and functions • Conditional statements and loops	• Three-dimensional plots • Solving transcendental equations • Numerical solutions of ODEs	• *Circuit Analysis* (concurrent) • *Calculus III* (Multivariable/vector calculus) (concurrent) • *Advanced Engineering Mathematics*
Module #3 Posted in the 11th week of the 13 week semester	• Numeric Integration • Additional Engineering Applications	• Visualizing Fourier series • Calculation of residues and poles of a complex function (used for circuit design) • Bode plots	• *Circuit Analysis* (concurrent) • *Signals and Systems* (future course) • *Advanced Engineering Mathematics*

Each module contained an introduction to the main Matlab concepts with worked examples and important command or code syntax. As well, the modules contained a series of rudimentary exercises to ensure the students get used to working with the Matlab environment to solve problems. The last part of each module contained a set of problems which were focused around a specific ECE application. These applications were drawn from the AEM course, one of the other courses they were taking in the same term (concurrent), or from a course they would take in a later semester. In all cases, the integration with other courses was made explicit so that students could see how that particular functionality of Matlab and the related concept within the AEM course was useful from an engineering perspective.

For the applications which were related to concepts in one of their concurrent courses, these were carefully timed so that they coincided with each other in both courses. In this way, the students were given the opportunity to immediately apply the new mathematical "tool" to an engineering problem. This "tool" consisted of the core mathematical concept which they learned about in the lectures and tutorials of the AEM course, and the numerical implementation that they learned through the Matlab modules. For example, the first module showed the students how to solve a set of simultaneous equations which was directly applicable to the multi-loop DC circuit problems which they were solving in their Circuit Analysis course at the same time. While in the last module the students learned how to determine the inverse Laplace transform of rational functions using the residue command in Matlab. This enabled them to work through s-domain circuit design problems that they had just seen in Circuit Analysis.

Online Matlab Quizzes

In order to encourage the students to work through the self-study modules they were asked to complete two 1-hour online quizzes, which each counted for 4.5% of their final course grade. These quizzes were implemented through the course management website and contained a combination of different types of questions, i.e., multiple choice, multiple answer, numeric answer, fill in the blank, etc. The questions required them to make used of Matlab to solve problems so they were more difficult than standard syntax memorization. A description of these two quizzes is given in Table 2.

Table 2. Description of Matlab Online Quizzes

Quiz	Coverage	Class Average
Quiz #1 Ran during the 5th week of the 13 week semester	Matlab Modules #1 and #2 Basic syntax, arrays/matrices, 2D and 3D plots, scripts and functions, solving sets of simultaneous equations, solving transcendental equations, solving first-order ODEs	72%
Quiz #2 Ran during the 13th week of the 13 week semester	All Three Matlab Modules Focus on applications: Bode plots, calculating Fourier series coefficients, numeric integration, and residue and pole calculations	82%

The students were given three days in which they had to complete the quiz, but once they started it they had to finish it within an hour. There was no supervision for these quizzes and they could complete them at any time within those three days. An effort was made to "individualize" the quizzes by creating pools of similar questions from which a student's questions were randomly selected. In addition, the questions which required numerical answers contained variables in the questions so each student's answer was uniquely generated when the question was selected for their quiz. Some sample questions are shown in Figures 1 – 3.

Figure 1. Sample Matlab Quiz #1 Question on Numeric Solutions of ODEs

For the initial value problem given below, match up the statements on the left with their correct answers on the right. Note, you can use the answers on the right more than once if needed.

$$\frac{dy}{dx} = \frac{4}{1+x^2}, \ y(a) = b$$

If the initial condition is y(0) = 0, then the particular solution has a value of y(1) approximately equal to...

If the initial condition is y(-3) = 0, then the curvature of the particular solution at x = -2 is...

If the initial condition is y(-3) = -4, then the curvature of the particular solution at x = 2 is...

If the initial condition is y(-5) = -6, then the slope of the particular solution at x = 3 is...

True/False: If the initial condition is y(-5) = -8, then the particular solution has a point of inflection somewhere between x = 8 and x = 10.

If the initial condition is y(0) = -5, then the particular solution has a value of y(3) approximately equal to...

A. 2.45
B. π
   (Pi)
C. True
D. Concave down
E. 1.5
F. Negative
G. Positive
H. False
I. Concave up
J. 0

Figure 2. Sample Matlab Quiz #2 Question on Numeric Integration

Use MATLAB to determine the value of the integral given by:

$$\int_a^b \frac{e^{3x}}{(x+4)^6} dx$$

with a = -3 and b = 3.

Figure 3. Sample Matlab Quiz #2 Question on Bode Plots

Consider the Bode plot of the magnitude of the transfer function given by:

$$H(j\omega) = \frac{10j\omega}{(j\omega+5)(j\omega+2)\left[(j\omega)^2+2j\omega+3\right]}$$

over the range

$$0.001\, rad\, /\, s \le \omega \le 1000\, rad\, /\, s$$

From the list of statements about this Bode plot given below select all those statements which are TRUE. Make sure you plot the magnitude of the transfer function using the decibel scale, i.e.,

$$20\log_{10}(|H(j\omega)|)$$

Note, a slope of 20 dB/decade means that as you increase w (omega) by a factor of 10 (i.e., you go from w = 10 rad/s to 100 rad/s), the value of the magnitude increases by 20 decibels (dB).

☐ Over this range of w (omega) the slope is constant at -20 dB/decade.

☐ Over this range of w (omega) the peak value of the transfer function is approximately -8.6 dB.

☐ Over this range of w (omega) the slope is always negative.

☐ For w = 0.01 rad/s, the slope of the magnitude curve is +20 dB/decade.

☐ Over this range of w (omega) the peak value of the transfer function is approximately 7 dB.

☐ The decibel value of the magnitude of the transfer function at w = 100 rad/s is -60 dB, i.e.,

$$20\log_{10}(|H(j100)|) = -60\, dB$$

☐ The decibel value of the magnitude of the transfer function at w = 100 rad/s is -100 dB, i.e.,

$$20\log_{10}(|H(j100)|) = -100\, dB$$

☐ Over this range of w (omega) the slope is always positive.

However, in speaking with some of the students it is clear that many of them worked together in groups to complete the quiz but it is not clear if this helped or hindered them. This is because some of them may simply have used their friend's results for their answers and not realized the subtle differences in the questions. Overall, the averages for the two quizzes were quite reasonable (72% and 82%), indicating that they were still a reasonable challenge. These were the only Matlab-related assessments for this component as the midterm and final exam did not include anything to do with Matlab.

Office Hours and Video Tutorials

Since there was no in-class lecture time devoted to discussing Matlab, a set of "office hours" were set up throughout the term, with a total of five 2-hour sessions being made available. In these sessions a course TA was present in one of the faculty's computer labs to answer any questions which a student may have had about the modules. Over the course of the term it seems that about 40% of the students in the course took advantage of these hours at one point or another.

The only other support that was provided directly to the students was a set of video tutorials which were posted about a week after each module was posted. These video tutorials discussed a select set of the exercises within each module and demonstrated how to complete these exercises in Matlab. The solutions to some of the application exercises were also presented, but the complete solutions for the exercises were never made available. By tracking the access to these posted videos it was found that roughly 85% of the students viewed part or all of these videos at one time or another during the term. During the term there were two significant spikes in the overall access of these which coincided with the timing of the online quizzes.

Survey Results and Statistical Analysis

In the last week of the term an online survey for the course was administered through the course management website. Part of this survey included questions dealing with the Matlab component for the course, and the results of these questions are summarized in Table 3. Of the 300 students registered in the course, 178 completed the survey (a 60% response rate).

Overall Results

A key observation from these results is that overall the Matlab component was well received by the students. Indeed, 65% of the class agreed that these modules were a useful introduction to Matlab, and 72% were happy that they were introduced to Matlab at this early stage of their four-year degree. When asked what the "Best part of the Matlab component was" many students made comments along these lines, as is summarized in Table 4. A lot of these comments dealt with the fact that they appreciated learning how to use this new analysis tool and how they had begun to use it for their other courses. However, the results were not overwhelmingly positive and there are some areas that can definitely be improved upon.

Lessons Learned

These results indicate that less than half of the respondents (43%) found that this Matlab component increased their overall understanding the core concepts within the AEM course. Since one of the main reasons for introducing this component was to provide some engineering context to these mathematical concepts, it is clear that future offerings of this component could improve in this area. The inclusion of well-designed model-eliciting activities would help in this respect[16]. As well, just about half of the respondents (49%) agreed that they were now comfortable with Matlab. Again, for this to be a successful addition to this AEM course this percentage needs to increase.

Table 3. Student Survey Results

Question	Student Response (5 – Strongly Agree, 4 – Agree, 3 – Neither Agree or Disagree, 2 – Disagree, 1 – Strongly Disagree)		Percent that Agrees or Strongly Agrees	Question Average
Overall, the modules were a useful introduction to Matlab.	Strongly Agree	9%	65%	3.44
	Agree	56%		
	Neither Agree nor Disagree	16%		
	Disagree	10%		
	Strongly Disagree	9%		
The Matlab modules were well organized and easy to follow.	Strongly Agree	6%	53%	3.26
	Agree	47%		
	Neither Agree nor Disagree	24%		
	Disagree	15%		
	Strongly Disagree	8%		
Now that I have completed these modules, I am reasonably comfortable with using Matlab to solve engineering problems.	Strongly Agree	3%	49%	3.19
	Agree	46%		
	Neither Agree nor Disagree	25%		
	Disagree	17%		
	Strongly Disagree	9%		
Since I am now familiar with Matlab I use it to solve problems on my own.	Strongly Agree	12%	53%	3.30
	Agree	41%		
	Neither Agree nor Disagree	21%		
	Disagree	16%		
	Strongly Disagree	10%		
The fact that the Matlab modules incorporated ideas from other courses made the process of learning Matlab more interesting.	Strongly Agree	6%	55%	3.33
	Agree	49%		
	Neither Agree nor Disagree	24%		
	Disagree	13%		
	Strongly Disagree	8%		
By using Matlab to solve engineering problems that were based on the concepts in MAT290 my overall understanding of these MAT290 concepts was increased.	Strongly Agree	6%	43%	3.16
	Agree	37%		
	Neither Agree nor Disagree	29%		
	Disagree	22%		
	Strongly Disagree	6%		

Table 3. Student Survey Results (continued)

Question	Student Response (5 – Strongly Agree, 4 – Agree, 3 – Neither Agree or Disagree, 2 – Disagree, 1 – Strongly Disagree)		Percent that Agrees or Strongly Agrees	Question Average
I found the posted video solutions for the Matlab modules to be very helpful in increasing my understanding on how to use Matlab.	Strongly Agree	25%	72% (of those that watched the videos)	3.86
	Agree	35%		
	Neither Agree nor Disagree	14%		
	Disagree	8%		
	Strongly Disagree	2%		
	Not Applicable. I never watched the posted videos.	16%		
I found the Matlab TA office hours to be very helpful in increasing my understanding on how to use Matlab.	Strongly Agree	5%	53% (of those that attended the office hours)	3.53
	Agree	16%		
	Neither Agree nor Disagree	12%		
	Disagree	5%		
	Strongly Disagree	1%		
	Not Applicable. I never attended the office hours.	60%		
The Matlab quizzes were a fair assessment of my ability to use Matlab.	Strongly Agree	4%	34%	2.75
	Agree	30%		
	Neither Agree nor Disagree	24%		
	Disagree	21%		
	Strongly Disagree	22%		
Overall, I am glad that I was introduced to Matlab at this early stage of the ECE program.	Strongly Agree	23%	72%	3.79
	Agree	49%		
	Neither Agree nor Disagree	18%		
	Disagree	5%		
	Strongly Disagree	5%		

From this experience and these survey results some important lessons have been learned.
1) *The students need more structured learning support*:
   Many of the student comments relating to the "Worst part of the Matlab component" dealt with the fact that it would have been better to have an introduction to Matlab covered in a few lectures or in extra tutorials. In the future, specific tutorials, video lectures, or interactive computer-based tutorials (such as Brian Daku's M-Tutor[17]) will be added to the course to give the students a more structured introduction to Matlab. The fact that students found the video tutorials and the TA office hours to be quite beneficial indicates that augmenting these supports will make a big difference in increasing the students' ability to use Matlab effectively. This is supported by the fact that those students who reported that they attended the office hours at some point in the term ended up with a higher average on the second online quiz. These results are summarized in Table 5. Since the second quiz covered more difficult applications of Matlab than the first quiz it makes sense that those students that

Table 4. Representative Samples of Student Comments

What is the best part of the Matlab component of this course?	What is the worst part of the Matlab component of this course?
Getting to know a tool and its features so that I can take advantage of it in future classes without struggling to learn it at that time.	The quizzes. Considering how many people worked with others, I felt penalized for following the rules.
Being able to graph 3-D plots and solve differential equations with ease. In addition, the videos were a great help in tackling some of the harder module problems.	That we had to learn the concepts on our own. If there were lectures about how to do the Matlab modules I believe that I would have had a better understanding of using Matlab at first.
It gave me a better understanding of the course material in a different way than the lecture and tutorial did.	It was hard to complete the questions in each module after following the instructions.
Learning to use a new tool to solve problems taught in a variety of courses during my second year.	The modules weren't completely descriptive on what to do. Like the hardest concept of the second module was covered in four lines. If I had not asked the TA in office hours, I would have had a terribly bad grade in the quizzes.

received extra help would have scored better on this quiz. The similarities in the averages for the midterm and the final exam suggest that these two groups each consist of a reasonable cross-section of students.

2) *Better assessment methods are needed*: The quizzes were listed as the "worst part of the Matlab component" by a very large percentage of the students. This view is also supported by the fact that only 34% of the students found them to be a fair assessment method. The online quizzes were a necessity for such a large group, and it was not logistically possible to have a single supervised session for all 300 students. Better methods could include having multiple supervised online quizzes, at the cost of the larger overhead of question preparation, or more individualized testing using TAs to conduct short oral exams..

3) *The organization and content of the modules needs to be improved*: Many students commented that they found the modules unclear at times and that the guided learning process was not gradual enough. In addition, the content of the modules could be improved upon to better contextualize the mathematical concepts within the AEM course.

Table 5. Averages for Students Who Did Not Attended/Did Attend Office Hours

Course Assessment	Average for students who did not attend TA office hours at any point (105 students)	Average for students who attended the TA office hours at some point (65 students)	p-value (two-tailed, unpaired) (* significant at a level of 0.05)
Matlab Quiz #1	73.8%	73.2%	0.813
Matlab Quiz #2	82.8%	87.6%	0.032*
Midterm	71.5%	70.6%	0.636
Final Exam	63.7%	61.6%	0.464

Conclusions

This paper has discussed a new Matlab-based component that has been incorporated into an advanced engineering mathematics course as part of the sophomore level electrical and computer engineering program. One of the main purposes for this component was to provide an opportunity for the students to see the course material in an engineering context through the numerical solution of "real problems". These problems were drawn from other ECE courses in an effort to create a more integrative experience. In addition, the component was designed to give the students a basic introduction to Matlab and it was assumed that they had no prior experience with this software.

The component consisted of three self-study modules, two online quizzes, a set of teaching assistant office hours, and a number of video tutorials. From the student survey results, the component was received quite positively, especially considering that this is the first year it has been offered. It was found that 65% of the class agreed that these modules were a useful introduction to Matlab, and 72% were happy that they were introduced to Matlab at this early stage of their degree. Yet the results also showed that only 49% of the class felt comfortable with Matlab after completing this component, and just 43% found that this component helped them to better understand the core mathematical concepts of the AEM course. Therefore, for future offerings of this component, efforts will be made to provide greater support in their learning of Matlab through additional in-class tutorials, video tutorials, and the use of more online or computer-based resources. As well, the self-study modules will be revised to provide a more gradual learning process and to better contextualize and integrate the core mathematical concepts. Lastly, the assessment plan will be modified so that the quizzes are more fair, through additional supervision, and transparent so that they can aid in their learning as opposed to strictly acting as a summative assessment.

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Appendix A: Third Matlab Self-Study Module - Numeric Integration and Engineering Applications

EXERCISES

In the previous two modules you have learned the basic interface of MATLAB, how to work with arrays, how to generate 2D and 3D plots, and how to write scripts and functions. In addition you have been exposed to a variety of ways in which MATLAB such as solving sets of simultaneous equations, solving for the roots of a polynomial equation, solving transcendental equations, and solving first-order ODEs numerically. This module will introduce you to the idea of numeric integration in addition to a number of applications which are specifically useful for electrical and computer engineering.

Review

At this point you should be reasonably comfortable with the essential functionality of MATLAB, which includes:

1) *Defining vectors and matrices*. For example, `A = [1 -5 6; 9 0 1; 2 9 0]` is a 3 x 3 matrix
2) *Matrix calculations*. For example, using the matrix operators (`+`, `-`, `*`, `/`, `inv`, etc.)
3) *Array calculations (element-by-element, or dot operators)*. For example, to calculate the expression $(e^x + 1)^2$, for the values of x ranging from -5 to 15, we could use the commands:
   ```
   >> x=linspace(-5,15,100) %x has 100 elements equally spaced from -5 to 15
   >> y=(exp(x)+1).^2  %the .^ operator is used to square each element
   ```
4) *Creating 2D and 3D plots*. Understanding how to use the commands `plot`, `subplot`, `mesh`, `meshc`, `meshgrid`, `ezplot`, `ezmeshc`, `figure`, `clf`, `hold`, `axis`, `xlabel`, `ylabel`, `legend`, `title`, `grid`, etc.
5) Writing, running, and debugging functions and scripts. Understanding the essential techniques of using scripts and functions, including the use of inline and anonymous functions.

Numerical Integration in MATLAB

A very useful function in MATLAB is `quadl`. This is one of a suite of functions which allow the user to evaluate a definite integral numerically. The functions `dblquad` and `triplequad` enable you to calculate double and triple integrals over square and rectangular regions, respectively.

All of these functions require the user to first define a function which defines the integrand. The built-in integrating function (e.g., `quadl`) calls this function as part of its calculations. This function can be a standard m-file function, an inline function, or an anonymous function.

For example, to evaluate the integral

$$\int_{-2}^{6} x^3 dx$$

One could first define the function an m-file as:

```
function [integrand]=myint(x)
integrand=x.^3;
```

Then using the following command, you can evaluate this definite integral as:

```
>> Q=quadl(@myint,-2,6)

Q =

    320
```

Which is the correct value, that can be verified by calculating:

```
>> 6^4/4-(-2)^4/4

ans =

    320
```

Alternatively, we could use an inline function to define our integrand, such as:

```
>> myint=inline('c*exp(j*pi*x)','x','c')

myint =

     Inline function:
     myint(x,c)  =  c*exp(j*pi*x)
```

In this case our integrand consists of two variables, c, and the variable of integration, x. To make sure the function quadl knows which variable is the variable of integration we would use the syntax:

```
Q=quadl(@(x)myint(x,2),0,1)

Q =

    0.0000 + 1.2732i
```

This command evaluates the integral:

$$\int_{0}^{1} 2e^{j\pi x} dx = \int_{0}^{1} 2\cos(\pi x)\, dx + j \int_{0}^{1} 2\sin(\pi x)\, dx$$

Exercises: *Numeric Integration*

1. Use the `quadl` function to evaluate the following integrals:

    a) $\int_1^3 \left(x^2 - \frac{1}{x^2} \right) dx$

    b) $\int_0^{\sqrt{3}} x^5 \sqrt{x^2 + 1}\, dx$

    c) $\int_{-\pi/3}^{\pi/3} (1 + x^2 - \cos x) dx$

    d) $\int_0^{2\pi} \frac{1}{(2+\cos \theta)^2} d\theta$

    e) $\int_0^{\pi/4} \sec x\, (2 \tan x - 5\sec x) dx$

2. The Gaussian function, $f(x) = e^{-x^2}$, is a function which shows up in many areas of engineering, most notably in probability theory which is a cornerstone of designing modern communication systems. A plot of the curve is shown below and from this it can also be seen why this is sometimes referred to as the "bell curve". The area under this curve is given by:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2 \int_0^{\infty} e^{-x^2} dx$$

This area turns out to be given by a very simple expression (although it is not so easy to prove analytically). Use MATLAB to determine the limiting value for this area using the `quadl` function. This limiting value has a simple relationship to one of math's most famous constants, can you figure out what this relationship is?

3. For this exercise we will be considering the periodic function plotted to the right. It is possible to express this periodic function in terms of the *Fourier series expansion* given by:

$$f(t) = \sum_{n=0}^{\infty} b_n \sin(n\omega_0 t)$$

where the coefficient can be calculated from:

$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} t \sin(n\omega_0 t) dt$$

with $\omega_0 = \frac{2\pi}{T_0}$. From these equations you can see that the Fourier series expansion of a periodic function consists of adding up sinusoidal terms which are all based upon the same fundamental frequency, ω_0, e.g., $\sin(\omega_0 t), \sin(2\omega_0 t), \sin(3\omega_0 t)$ .... Each of these terms are called the harmonics of the original sinusoid, $\sin(\omega_0 t)$. For example, $\sin(3\omega_0 t)$ is the 3rd harmonic term. Each of these harmonics are scaled by the constant b_n, and this constant tells us how much of the nth harmonic contributes to the original periodic signal (such as the ramp function in this case).
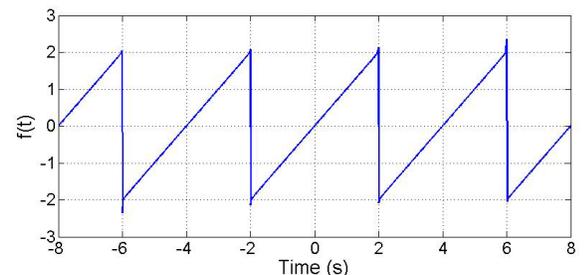
    a) To determine the Fourier series expansion for the periodic ramp function shown above, calculate the coefficients for the first 21 harmonics. This can be done by evaluating the integral given by $b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} t \sin(n\omega_0 t) dt$, for $T_0 = 4$ s for the integers n from 0 to 20.

b) Use your results to plot the function given by the series expansion $f(t) = \sum_{n=0}^{20} b_n \sin(n\omega_0 t)$, over the range $-8\,\text{s} \le t \le 8\,\text{s}$. This is the Fourier series expansion for the periodic ramp function using the first 21 terms of the expansion. How close does your plot resemble a ramp function over this range?

c) Repeat your plot, but plot the four functions, $f_k(t) = \sum_{n=0}^{k} b_n \sin(n\omega_0 t)$, for $k = 1,5,10,15$, on the same plot. As you add more terms to the expansion how does the Fourier series expansion relate to the ramp function over this range.

d) Use the `stem` plot command to plot the 21 coefficients, b_n, versus the frequency vector given by $\omega = n\omega_0$. Which harmonic contributes the most to the Fourier series expansion of this periodic ramp function? Which harmonic contributes the least?

Engineering Applications: Residue and Pole Calculations

One of the challenging parts of working with the Laplace transform and the s-domain is the fact that the inverse transform often requires partial-fraction expansion to find the poles of the function and the residues of each pole. MATLAB's built in function `residue` rapidly calculates the location of the poles and their associated residues, which enables the inverse Laplace transform to be more quickly found.

For example, given the s-domain function of $F(s) = \frac{120}{s^2+3s+2}$, the poles and residues can be found from

```
>> [r,p]=residue([120],[1,3,2])
r =
   -120
    120
p =
     -2
     -1
```

Observe how the coefficients of s for the numerator and denominator of the rational function $F(s)$, are specified as two input vectors to the `residue` function. From this the inverse is then given by:
$$f(t) = (-120e^{-2t} + 120e^{-t})u(t)$$

As another example, consider the s-domain function $G(s) = \frac{12s+36}{s(s^2+4s+5)} = \frac{12s+36}{s^3+4s^2+5s}$, whose poles and residues are found from:

```
>> [r,p]=residue([12,36],[1,4,5,0])
r =
   -3.6000 + 1.2000i
   -3.6000 - 1.2000i
    7.2000
p =
   -2.0000 + 1.0000i
   -2.0000 - 1.0000i
         0
```

Given this, we can call upon the inverse transform,

$$L^{-1}\left\{\frac{A}{s+\alpha-j\beta}+\frac{\bar{A}}{s+\alpha+j\beta}\right\} = 2|A|e^{-\alpha t}\cos\left(\beta t+\theta\right)$$

where $A = |A|\angle\theta$ is the complex-valued residue of the pole $p_1 = -\alpha+j\beta$, and $\bar{A} = |A|\angle-\theta$ is the complex conjugate of A. Therefore, we can used the commands:

```
>> abs(r)
ans =
    3.7947
    3.7947
    7.2000
>> angle(r)*180/pi
ans =
  161.5651
 -161.5651
        0
```

To determine that the inverse of this function $G(s)$ is:

$$g(t) = [7.2 + 7.58e^{-2t}\cos(t + 161.6°)]u(t)$$

Exercises: *Residue and Pole Calculations*

1. Find the inverse Laplace transforms of the following *s*-domain functions:

   a) $V(s) = \frac{8(s+3)}{(s+1)^2}$

   b) $F(s) = \frac{10s^3}{(s^2+4)(3s^2+4s+4)}$

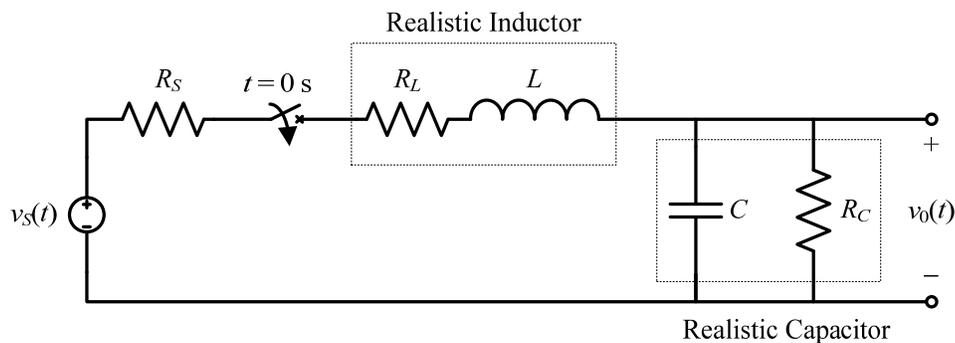   c) $I(s) = \frac{2s+1}{(s+1)^3}$

   d) $V(s) = \frac{20}{3}\frac{s+2}{(s^2+3s+3)}$

   e) $I(s) = \frac{5}{7s^3+31s^2+46s+24}$

2. *Circuit Design*
   Consider the circuit below:

Realistic Inductor

Realistic Capacitor

It can be proved that the output voltage is governed by the differential equation:

$$v_0'' + \left(\frac{R_S + R_L}{L} + \frac{1}{R_C C}\right)v_0' + \left(\frac{R_S + R_L + R_C}{R_C LC}\right)v_0 = \frac{1}{LC}v_S(t)$$

Now, consider the case in which this circuit is initially uncharged (zero-state), $C = 1\,\mu F$, $L = 10\,H$, $R_S = 50\,\Omega$, $R_C = 25\,k\Omega$, and the input source is given as $v_S(t) = 5\cos(100t)$. The resistance of the inductor can be changed by adding in more resistance in series with this inductor, and for practical reasons it must lie within the range of $1\,k\Omega \le R_L \le 10\,k\Omega$.

a) Within this given range of R_L, for what part of this range would the response have the form given by:
$$v_0(t) = V_{01}\cos(100t + \theta_1) + V_{02}e^{-\alpha t}\cos(\omega_2 t + \theta_2)$$
where $V_{01}, V_{02}, \theta_1,$ and θ_2 are constants, and α and ω_2 are positive constants. *Hint*: To do this, you will have to find the s-domain expression for the solution $v_0(t)$, and then write a function which will allow you to determine the residues and poles of the s-domain function $V_0(s)$ for different values of R_L. Then if you plot the real and imaginary parts of these residues and poles as a function of R_L, then you can see how the time domain response will change as R_L changes.

b) Within this given range of R_L, for what part of this range would the response have the form given by:
$$v_0(t) = V_{01}\cos 100t + V_{02}e^{-\alpha_1 t} + V_{03}e^{-\alpha_2 t}$$
where $V_{01}, V_{02},$ and V_{03} are constants, and α_1 and α_2 are positive constants.

c) What should R_L be if the response should be
$$v_0(t) = 4.38\cos(100t - 127.1°) - 0.798e^{-476t} + 3.44e^{-212t}\ \text{V}$$

d) What should R_L be if the response should be
$$v_0(t) = 5.51\cos(100t - 96.7°) + 1.77e^{-53.3t}\cos(312t + 68.5°)\ \text{V}$$

Engineering Applications: Bode Plots

In your Circuits Analysis course you will soon learn about how useful the concept of Bode plots can be used to characterize a circuit's behavior. These plots are a graphical representation of how the circuit responses to sinusoidal signals of different frequencies. As an example consider the simple circuit shown on the left, along with its Bode plot on the right:
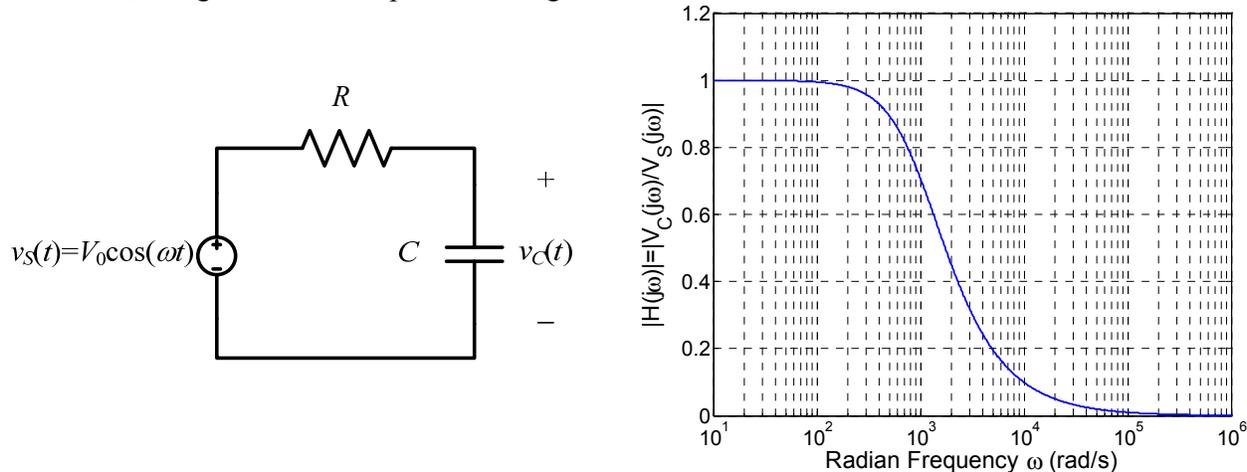
What the Bode plot shows is how the output voltage, $v_C(t)$, relates to the input voltage, $v_S(t)$. So, this plot indicates that when the sinusoidal input has a low frequency ($\omega < 10^2 = 100$ rad/s), the output voltage is the same as the input voltage (i.e., the ratio is equal to 1). As the frequency of the source increases, eventually the output voltage is close to zero ($\omega > 10^5 = 100,000$ rad/s). This makes intuitive sense, because at low frequencies a capacitor is essentially an open circuit, so $v_C(t) \approx v_S(t)$. At high frequencies a capacitor looks like a short circuit so, $v_C(t) \approx 0$.

To create the Bode plot you first need to find the ratio of the output quantity to the input quantity in the *frequency domain*. This can be done by first finding the ratio in the *s*-domain, which is (with zero initial conditions):

$$V_C(s) = V_S(s)\left[\frac{\frac{1}{sC}}{R+\frac{1}{sC}}\right] = V_S(s)\left[\frac{1}{RCs+1}\right] \quad \text{which gives} \quad H(s) = \frac{V_C(s)}{V_S(s)} = \left[\frac{1}{RCs+1}\right]$$

If we let $s = j\omega$, then we get this ratio in terms of the frequency of the source. This is call the *transfer function*, and is given by:

$$H(j\omega) = \frac{V_C(j\omega)}{V_S(j\omega)} = \left[\frac{1}{RCj\omega + 1}\right]$$

Since this is a complex-valued function it must be plotted in terms of its magnitude and phase. Typically, the magnitude of this transfer function is the most useful, $|H(j\omega)|$ and this is what is graphed in the Bode plot above.

In MATLAB, this plot can be generated using the following set of commands:
```
>> R=1000;C=1e-6;w=2*pi*logspace(0,6,1000);
>> H=1./(R*C*j*w+1);
>> semilogx(w,abs(H));
```

Here the command `logspace` has been used to create a wide range of ω, $10^0 \le \omega \le 10^6$, and a `semilogx` plot was used to create the linear y-axis and the logarithmic x-axis.

Exercises: *Bode Plots*

1. Plot the magnitude of the frequency-domain functions given below.

   a) $H(j\omega) = \frac{2j\omega+1}{(j\omega+1)^3}$, over the range of $10^{-3} \le \omega \le 10^2$.

   b) $G(j\omega) = \frac{10^4(j\omega+2)}{(j\omega+10)(j\omega+100)}$, over the range of $10^{-3} \le \omega \le 10^3$. What type of response would this Bode plot describe? Consider how input signals of different frequencies would appear at the output of circuit described by this transfer function.

   c) $G(j\omega) = \frac{10^4(j\omega+2)}{(j\omega+10)(j\omega+100)}$, over the range of $10^{-3} \le \omega \le 10^3$.

2. *Circuit Design*
   For the circuit considered above, the transfer function can be written as:

   $$\frac{V_0(j\omega)}{V_S(j\omega)} = H(j\omega) = \frac{1}{LC} \frac{1}{(j\omega)^2 + j\omega\left(\frac{R_S + R_L}{L} + \frac{1}{R_C C}\right) + \left(\frac{R_S + R_L + R_C}{R_C LC}\right)}$$

   Plot this transfer function for the two values of R_L that you found in parts c) and d) of Exercise #2 for residues and poles. Explain the main difference between these two Bode plots.